ISRAEL JOURNAL OF MATHEMATICS 77 (1992), 33–54

SUBSEQUENCE POINTWISE ERGODIC THEOREMS FOR OPERATORS IN L^p

BY

ROGER L. JONES*

Department of Mathematics
2219 N. Kenmore, Chicago, IL 60614, USA

AND

JAMES OLSEN

Department of Mathematics, North Dakota State University
Fargo, ND 58105, USA

ABSTRACT

In this paper certain subsequence ergodic theorems which have previously been known in the case of measure preserving point transformations are extended to Dunford–Schwartz operators, positive isometries, and power bounded Lamperti operators.

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a probability space. Let T denote a linear operator of $L^p = L^p(X, \mathcal{F}, \mu)$, $1 \leq p \leq \infty$, and $\{n_k\}_{k=0}^{\infty}$ an increasing sequence of positive integers. In this paper we will be concerned with the almost sure convergence of the averages

$$\frac{1}{N}\sum_{k=0}^{N-1} T^{n_k}(f)(x) \tag{1.1}$$

for $f \in L^p$. In some cases we will assume T is a linear operator on L^p for a fixed p, and in other cases we will assume T is a linear operator on L^p for all p in

* Partially supported by NSF Grant DMS-8910947.
Received March 9, 1989 and in revised form May 23, 1991

an interval. Such "subsequence ergodic theorems" have been studied by many authors, notably Baxter and Olsen [4], Bellow and Losert [8], [9], Bourgain [10], [11], [12], and Wierdl [29].

If the operator T has the form $T(f)(x) = f o \tau(x)$, where τ is a measure preserving point transformation, we will say that T is induced by the measure preserving point transformation τ. If $\|T\|_1 \leq 1$ and $\|T\|_\infty \leq 1$, we will say that T is a Dunford–Schwartz operator. By the density of the sequence $\{n_k\}$ we will mean

$$\lim_{N\to\infty} \frac{|\{n_k\}_{k=0}^{\infty} \cap [1, N]|}{N},$$

if this limit exists. Here $|A|$ is the cardinality of the set A.

To avoid certain measure theoretic difficulties, we will assume throughout the paper that $(X, \mathcal{F}, \mu)$ is a Lebesgue space.

In Section 2 of this paper we study maximal inequalities associated with averages given by (1.1). Section 3 uses the results of Section 2 to prove a.e. convergence of these averages for certain classes of sequences, and general operators such as Dunford–Schwartz operators and power bounded Lamperti operators. In particular, for these more general operators, we will have convergence for the block sequences considered by Bellow and Losert [8].

In [4] Baxter and Olsen show that if $\{n_k\}_{k=0}^{\infty}$ has positive density, and if the averages given by (1.1) converge for each T induced by a measure preserving point transformation, then this same limit exists almost surely for T a Dunford–Schwartz operator. Section 4 of this paper generalizes their result to the case of zero density. In the case of zero density a norm inequality for the associated maximal function, such as obtained in Section 2, is also needed. (In the case of positive density the necessary norm inequality for the maximal function always holds for trivial reasons.)

For special sequences $\{n_k\}_{k=0}^{\infty}$ that have zero density, Bourgain, in a sequence of papers, [10], [11], [12], obtains the existence of the limit for averages of the form (1.1) for all T induced by measure preserving point transformations. In particular he obtains this convergence for $n_k = k^2$, for all $f \in L^p$, $p > 1$. The results of Sections 2 and 4 will establish conditions under which the convergence of (1.1) for operators on L^p, $p > 1$, which are induced by measure preserving point transformations, implies the convergence of (1.1) for Dunford–Schwartz operators. In particular, combining our results with those of Bourgain, we will show that for $f \in L^p$, $p > 1$, and for T a Dunford–Schwartz operator, a.s.

convergence takes place along $\{n_k\}_{k=0}^{\infty}$ where $n_k = k^2$. Using the results of Bourgain [11] and Wierdl [29] we obtain convergence for such operators when n_k is the kth prime.

2. Dominated Estimates

In this section we will obtain several "strong type" norm inequalities that we will use in later sections. Since we will want to use these inequalities in more than one context, we will generalize our notation. In this section, a "subsequence" will be a function $n : N \times N \to N$ such that

(a) $n(k_1, \ell_1) > n(k_2, \ell_2)$ if $\ell_1 > \ell_2$

and

(b) $n(k_1, \ell) \geq n(k_2, \ell)$ if $k_1 \geq k_2$.

Our usual subsequence $\{n_\ell\}_{\ell=0}^{\infty}$ will then be the subsequence $n(k, \ell) = n_\ell$. for all k.

Let T be a linear operator on L^p. We will define the operators

$$A_{k,\ell}(f)(x) = \frac{1}{\ell} \sum_{j=0}^{\ell-1} T^{n(k,j)}(f)(x), \tag{2.1}$$

$$M_{K,L}(f)(x) = \sup_{\substack{k \leq K \\ \ell \leq L}} |A_{k,\ell}(f)(x)| \tag{2.2}$$

and

$$M(f)(x) = \sup_{k,\ell} |A_{k,\ell}(f)(x)|. \tag{2.3}$$

Note that it is possible for $M(f)$ to be $+\infty$.

If $T(f)(x) = w(x)f(\tau x)$ where τ is a point transformation, $\tau : X \to X$, we will say that T is induced by the point transformation τ with weight function w, generalizing the measure preserving case. If the operator T is induced by a point transformation τ, and τ is aperiodic, we will say that T is aperiodic.

Operators induced by point transformations are more general than might be expected. In 1958 J. Lamperti [23] proved that if T is a linear operator on a space $L^p(X)$ for some $p \neq 2$, and $\|Tf\|_p = \|f\|_p$, then there is a function $w(x)$ and a transformation S such that $Tf(x) = w(x)Sf(x)$, where the transformation S is induced by a "regular set isomorphism". In other words, $S(\chi_A) = s(A)$ where s satisfies $s(X - A) = s(X) - s(A)$, $s\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} s(A_n)$ for disjoint A_n, and $\mu(sA) = 0$ if and only if $\mu(A) = 0$. Assuming suitable regularity conditions on $(X, \mathcal{F})$ (X a complete separable metric space and $\mathcal{F}$ the σ-algebra of Borel sets) the set isomorphism s can be shown to be induced by a non-singular point transformation. (See [15] pages 453 and following for more details.) See also [14] and [27]. For the case $p = 2$ see [5]. Thus isometries on $L^p(X)$ are induced by point transformations.

We will say that T admits a dominated estimate along $n(k,j)$ in L^p with constant c if $\|Mf\|_p \le c\|f\|_p$, where T is the operator used in (2.1) and (2.3).

In this section, we generalize the construction of Jones [19] to show that if there exists a single aperiodic positive invertible isometry S, induced by a point transformation σ, such that S admits a dominated estimate along $n(k,\ell)$ with constant c, then T admits a dominated estimate along $n(k,\ell)$ for all aperiodic positive invertible isometries T. This in turn will imply that T admits a dominated estimate in L^p for all Dunford–Schwartz operators T.

2.1 THEOREM: *Let S and T be positive invertible isometries of $L^p(Y, \mathcal{S}, \nu)$ and $L^p(X, \mathcal{F}, \mu)$ respectively, $1 < p < \infty$. If S is aperiodic and admits a dominated estimate in L^p with constant c along $n(k,j)$, then T also admits a dominated estimate along $n(k,j)$ in L^p with constant c.*

Proof: Let T and S be induced by the non-singular point transformations τ and σ respectively. Let $\{w_n\}$ and $\{w'_n\}$ be the functions such that $T^n(f)(x) = w_n(x)f(\tau^n x)$ and $S^n(f)(y) = w'_n(y)f(\sigma^n y)$. First assume that τ is aperiodic, then given $\epsilon > 0$ and N, there exist disjoint sets A_1, $A_2, \ldots$, A_N such that $\tau : A_k \to A_{k+1}$, $k = 1, 2, \ldots, N-1$, and $\mu(X - \bigcup_{i=1}^{N} A_i) < \epsilon$. (See [18] page 71 for a proof in the measure preserving case. The details necessary here can be found in [26] in the case when the measure is non-atomic. If the space contains atoms, small but straightforward modifications are required.) Further, from the construction of these sets, we see that if $N = m^2$, the sets A_{m^2-m+1}, A_{m^2-m+2}, $\ldots$, $A_{N=m^2}$ can be taken to have total measure less than $1/m$. (See [26].) We will say the sets $A_1, \ldots, A_N$ form a Rohlin–Kakutani tower for τ with height N and error less than ϵ.

Let $f \in L^p$. Fix K and L. Let $M_{K,L}^T$ be defined by

$$M_{K,L}^T(f)(x) = \sup_{\substack{k\le K\\ \ell\le L}} |A_{k,\ell}^T(f)(x)|$$

where $M_{K,L}^T$ and $A_{k,\ell}^T$ denote the dependence on T of the operators (2.1) and (2.3). Choose δ such that

$$\int_E |M_{K,L}^T(f)(x)|^p d\mu < \epsilon \qquad \text{if } \mu(E) < \delta.$$

Let m be an integer such that $1/m < \delta/2$, and

$$m > \sup_{\substack{k\le K\\ \ell\le L}} n(k,\ell).$$

Let $A_1, A_2, \cdots, A_{m_2}$ be the sets of a Rohlin–Kakutani tower for τ, with error less than $\delta/2$, and such that $\mu(\bigcup_{k=m^2-m+1}^{m^2} A_k) < \delta/2$. Let $B_1, B_2, \ldots, B_{m^2}$ be the sets of a Rohlin–Kakutani tower for σ. Let β be a constant such that $\mu(A_1) = \beta\nu(B_1)$. Then A_1 has the same measure as B_1 with respect to the measure $\beta\nu$. Let ℓ be a 1-1 measure preserving transformation from A_1 onto B_1. If $\operatorname{supp}(f) \subset A_1$, define the mapping H by $H(f)(y) = f(\ell^{-1}y)\beta^{1/p}$. Then H maps $L^p(A_1)$ to $L^p(B_1)$, and in fact

$$\begin{aligned}\int_{B_1} |Hf(y)|^p d\nu &= \int_{B_1} |f(\ell^{-1}y)\beta^{1/p}|^p d\nu\\ &= \int_{B_1} |f(\ell^{-1}y)|^p \beta d\nu\\ &= \int_{A_1} |f(x)|^p d\mu.\end{aligned}$$

Let $A = \bigcup_{k=1}^{m^2} A_k$, $B = \bigcup_{k=1}^{m^2} B_k$, and extend $H : L^p(A) \to L^p(B)$ as follows:

First extend ℓ from $A_1 \to B_1$ to $\mathcal{L} : A \to B$ by $\mathcal{L}(x) = (\sigma^k \ell \tau^{-k})(x)$ for $x \in A_k$, $k = 1, 2, \ldots, m^2$. Now define

$$H(f)(y) = f(\mathcal{L}^{-1}y)\frac{w_k(\tau^{-k}\mathcal{L}^{-1}y)}{w_k'(\sigma^{-k}y)}\beta^{1/p}.$$

We note that $\operatorname{supp}(Hf) \subset B_k$ if $\operatorname{supp}(f) \subset A_k$, and that

$$w_{m+n}(x) = w_n(\tau^m x) w_m(x),$$

with a similar equation for $\{w'_n\}$ and σ.

If $\mathrm{supp}(f) \subset A_k$ then

$$\begin{aligned}
\int |f(x)|^p d\mu &= \int |f(x)\chi_{A_k}(x)|^p d\mu \\
&= \int |f(\tau^k x)\chi_{A_k}(\tau^k x)|^p w_k^p(x) d\mu \\
&= \int_{A_1} |f(\tau^k x)|^p w_k^p(x) d\mu \\
&= \int_{B_1} |f(\tau^k \ell^{-1} y)|^p w_k^p(\ell^{-1} y)\beta d\nu \\
&= \beta \int_{B_1} |f(\tau^k \ell^{-1} y)|^p w_k^p(\ell^{-1} y)\chi_{B_1}(y) d\nu \\
&= \beta \int |f(\tau^k \ell^{-1}\sigma^{-k} y)|^p w_k^p(\tau^{-k}\tau^k \ell^{-1}\sigma^{-k} y)\chi_{B_1}(\sigma^{-k} y) w'_{-k}(y) d\nu \\
&= \beta \int |f(\mathcal{L}^{-1} y)|^p w_k^p(\tau^{-k}\mathcal{L}^{-1} y)\chi_{B_k}(y) w'_{-k}(y) d\nu \\
&= \beta \int_{B_k} |f(\mathcal{L}^{-1} y)|^p \frac{w_k(\tau^{-k}\mathcal{L}^{-1} y)^p}{w'_k(\sigma^{-k} y)^p} d\nu
\end{aligned}$$

since $w'_{-k}(y) = (w'_k(\sigma^{-k} y))^{-1}$. Therefore,

$$\int_{A_k} |f|^p d\mu = \int_{B_k} |Hf|^p d\nu$$

and from this we have

$$\int_A |f(x)|^p d\mu = \int_B |H(f)(y)|^p d\nu.$$

Let $f \geq 0$. Fix $x \in A_r \cap \mathrm{supp}(M_{K,L}^T f(x))$. Then there exist integers $k \leq K$ and $\ell \leq L$ such that

$$\begin{aligned}
M_{K,L}^T f(x) &= \frac{1}{\ell} \sum_{j=0}^{\ell-1} T^{n(k,j)}(f)(x) \\
&= \frac{1}{\ell} \sum_{j=0}^{\ell-1} f(\tau^{n(k,j)} x) w_{n(k,j)}(x).
\end{aligned}$$

If $y = \mathcal{L}x$, then

$$
\begin{aligned}
A^S_{k,\ell}(Hf)(y) &= \frac{1}{\ell}\sum_{j=0}^{\ell-1} S^{n(k,j)}(Hf)(y)\\
&= \frac{1}{\ell}\sum_{j=0}^{\ell-1}(Hf)(\sigma^{n(k,j)}y)w'_{n(k,j)}(y)\\
&= \frac{1}{\ell}\sum_{j=0}^{\ell-1} f(\mathcal{L}^{-1}\sigma^{n(k,j)}y)\frac{w_{r+n(k,j)}(\tau^{-(r+n(k,j))}\mathcal{L}^{-1}\sigma^{n(k,j)}y)}{w'_{r+n(k,j)}(\sigma^{-(r+n(k,j))}\sigma^{n(k,j)}y)}\\
&\quad\times w'_{n(k,j)}(y)\beta^{1/p}\\
&= \frac{1}{\ell}\sum_{j=0}^{\ell-1} f(\mathcal{L}^{-1}\sigma^{n(k,j)}y)\frac{w_{r+n(k,j)}(\tau^{-(r+n(k,j))}\mathcal{L}^{-1}\sigma^{n(k,j)+r}\sigma^{-r}y)}{w'_{r+n(k,j)}(\sigma^{-r}y)}\\
&\quad\times w'_{n(k,j)}(y)\beta^{1/p}\\
&= \frac{1}{\ell}\sum_{j=0}^{\ell-1} f(\mathcal{L}^{-1}\sigma^{n(k,j)}y)\frac{w_{r+n(k,j)}(\mathcal{L}^{-1}\sigma^{-r}y)}{w'_{r+n(k,j)}(\sigma^{-r}y)}w'_{n(k,j)}(y)\beta^{1/p}\\
&= \frac{1}{\ell}\sum_{j=0}^{\ell-1} f(\tau^{n(k,j)}x)\frac{w_{r+n(k,j)}(\tau^{-r}x)}{w'_{r+n(k,j)}(\sigma^{-r}\mathcal{L}\,x)}w'_{n(k,j)}(\mathcal{L}\,x)\beta^{1/p}\\
&= \frac{1}{\ell}\sum_{j=0}^{\ell-1} f(\tau^{n(k,j)}x)\frac{w_r(\tau^{-r}x)}{w'_r(\sigma^{-r}\mathcal{L}\,x)}\frac{w_{n(k,j)}(x)}{w'_{n(k,j)}(\mathcal{L}\,x)}w'_{n(k,j)}(\mathcal{L}\,x)\beta^{1/p}\\
&= \frac{w_k(\tau^{-r}x)}{w'_k(\sigma^{-r}\mathcal{L}\,x)}\beta^{1/p}\frac{1}{\ell}\sum_{j=0}^{\ell-1} f(\tau^{n(k,j)}x)w_{n(k,j)}(x)\\
&= H(M^T_{K,L}f)(x).
\end{aligned}
$$

From this we see that $M^S_{K,L}(Hf)(y) \geq H(M^T_{K,L}f)(y)$. Thus we have

$$
\begin{aligned}
\|M^T_{K,L}f\|^p_p &= \int_X |M^T_{K,L}f(x)|^p d\mu\\
&\leq \int_A |M^T_{K,L}f(x)|^p d\mu + \epsilon.
\end{aligned}
$$

However we have shown that on A, H preserves L^p norms, hence we have

$$
\begin{aligned}
\int_A |M^T_{K,L}f(x)|^p d\mu + \epsilon &\leq \int_A |H(M^T_{K,L}f)(y)|^p d\nu + \epsilon\\
&\leq \|M^S_{K,L}(Hf)\|^p_p + \epsilon\\
&\leq c\|H(f)\|^p_p + \epsilon\\
&\leq c\|f\|^p_p + \epsilon.
\end{aligned}
$$

Let $\epsilon \to 0$, and then take supremums over K and L to complete the proof in the case τ aperiodic. If τ is periodic with period d, the only modification needed to make the argument work is to replace the sets $A_1, A_2, ..., A_{m^2}$ by disjoint sets $A_1, A_2, ..., A_d$ where $x \in A_1$ implies $\tau^k(x) \in A_k$, $k = 1, 2, ..., d$, and $\tau^{d+1}(x) = x$. Using the fact that τ, and hence τ^{-1}, is periodic, the map $\mathcal{L}(x) = (\sigma^k \ell \tau^{-k})(x)$ can still be used to extend $H : L^p(A_1) \to L^p(B_1)$ to $H : L^p(\bigcup_{k=1}^{m^2} A_k) \to L^p(\bigcup_{k=1}^{m^2} B_k)$. The remainder of the proof is the same with minor modifications based on the fact that $w_{n+d}(x) = w_n(x)$. ∎

2.2 COROLLARY: *Let T be dominated by a positive contraction of L^p, p fixed, $1 < p < \infty$. If there exists an aperiodic positive invertible isometry that admits a dominated estimate along $n(k,j)$ with constant $c > 1$, then so does T.*

Proof: Since T is dominated by a positive contraction we may assume that T is positive. By [3] we may assume that there exists a positive invertible isometry Q and a conditional expectation operator E such that $T^n = EQ^n$ for $n = 1, 2, \ldots$. Note that because E is positive, we have $M^T f \le EM^Q f$. Since E is positive and a contraction of all L^p spaces, we may assume that T is a positive invertible isometry. The corollary now follows from Theorem 2.1. ∎

Another family of operators on L^p spaces are the Lamperti operators studied by Kan [21]. A Lamperti operator is a linear operator that separates supports. If $T : L^p \to L^p$ is a linear operator with the property that there exists a constant b such that $\|T^n\|_p \le b$ for all n, then we say that T is power bounded with power bound b. We now extend Corollary 2.2 to power bounded Lamperti operators.

2.3 COROLLARY: *Let $T : L^p \to L^p$, p fixed, $1 < p < \infty$, be a power bounded Lamperti operator with power bound b. If there exists an aperiodic positive invertible isometry $S : L^p \to L^p$ such that S admits a dominated estimate along $n(k,j)$ with constant c, then T admits a dominated estimate along $n(k,j)$ with constant bc.*

Proof: By Corollary 4.1 of [21], there exists a positive Lamperti contraction $\bar{T}$ such that $|T^n f| = b\bar{T}^n |f|$. The corollary now follows from Corollary 2.2. ∎

2.4 COROLLARY: *Fix p, $1 < p < \infty$, and let $\{n_k\}$ be an increasing sequence of positive integers such that every T induced by a measure preserving transformation admits a dominated estimate in L^p along $\{n_k\}$. Then operators that are dominated by positive contractions, and power bounded Lamperti operators* *of L^p, admit a dominated estimate in L^p along $\{n_k\}$. In particular, Dunford–Schwartz operators admit a dominated estimate in L^p along $\{n_k\}$.*

Proof: In the notation of this section, $\{n_k\}$ corresponds to the subsequence defined by $n(k,j) = n_j$ for all k. Note that Dunford–Schwartz operators are dominated by positive contractions (the linear modulus). See [22]. The result now follows by an application of the above theorem and corollaries. ∎

Bourgain ([10] [11] [12]) has shown that for $n_k = k^2$, or more generally for $n_k = k^t$, t integer, for $p > 1$, T admits a dominated estimate in L^p along $\{n_k\}$ when T is induced by a measure preserving transformation. Thus we now have that all the types of operators thus far considered admit a dominated estimate in L^p along $\{n_k\}$. In particular, if T is Dunford–Schwartz, T admits a dominated estimate in L^p, $1 < p < \infty$ along $\{n_k\}$ for $n_k = k^2$ or more generally k^t.

3. Moving Averages and Block Sequences

The moving averages considered by Bellow, Jones and Rosenblatt [7] are averages of the form

$$A_k f(x) = \frac{1}{\ell_k}\sum_{i=0}^{\ell_k - 1} f(\tau^{n_k+i}x) = \frac{1}{\ell_k}\sum_{i=0}^{\ell_k-1} T^{n_k+i}(f)(x)$$

where T is induced by the measure preserving transformation τ and $\{n_k, \ell_k\}$ is a sequence of pairs of positive integers. In [7] conditions are investigated under which

$$\lim_{k\to\infty} A_k(f)(x) \text{ exists a.s. for } f \in L^p, \quad 1 \le p < \infty.$$

In our context, $A_k f(x) = A_{k,\ell} f(x)$ where

$$n(k,j) = n_k + j,\ 0 \le j < \ell_k, \qquad \text{and} \qquad n(k,j) = 0 \quad \text{if } j > \ell_k - 1.$$

Theorem 1 of [7] shows that if (n_k, ℓ_k) satisfies a certain growth condition, then $A_k f(x)$ converges for all f in L^1 and hence in L^p, and that, for $p > 1$, T admits a dominated estimate in L^p along $n(k,j)$. It is also shown that if $\lim_{k\to\infty} A_k f(x)$ exists a.s. for all $f \in L^p$ for some p, $1 < p < \infty$ then $n(k,j)$ satisfies the growth condition. See [7] for a precise statement of the growth condition, and examples of sequences that satisfy it. Note that combining the results of [7] with Theorem 2.1 we have the following:

3.1 THEOREM: *Let T be a power bounded Lamperti operator, or an operator dominated by a positive contraction on L^p, $p > 1$. Let $\{n_k, \ell_k\}$ be a double sequence of positive integers satisfying the growth condition of [7], and with $\ell_{k+1} > \ell_k$. Then*

$$\lim_{k\to\infty} \frac{1}{\ell_k} \sum_{i=0}^{\ell_k - 1} T^{n_k+i}(f)(x)$$

exists a.s. for all $f \in L^p$, $p > 1$.

Proof: First we show that we have convergence for a dense class of L^p functions. Let $g \in L^\infty$, and h such that $Th = h$. We will consider the function $f = g - Tg + h$, and note that the set of such f is dense in L^p, $p > 1$, by the Mean Ergodic Theorem [24]. For such an f we have

$$\lim_{k\to\infty} \frac{1}{\ell_k} \sum_{j=0}^{\ell_k - 1} T^{n_k+j}(f)(x) = \lim_{k\to\infty} h + \frac{T^{n_k}g - T^{n_k+\ell_k}g}{\ell_k} .$$

To see that this limit exists, we now show that

$$\lim_{k\to\infty} \frac{T^{n_k}(g)(x)}{\ell_k} = 0 \quad \text{for a.e. } x.$$

For any sequence $\{n_k\}_{k=1}^\infty$, since $\{\ell_k\}_{k=1}^\infty$ is increasing, we have

$$\int \sum_{k=1}^\infty \left| \frac{T^{n_k}(g)(x)}{\ell_k} \right|^p d\mu \le \sum_{k=1}^\infty \frac{\|T^{n_k}(g)\|_p^p}{\ell_k^p} \le c^p \|g\|_p^p \sum_{k=1}^\infty \frac{1}{k^p} < \infty.$$

Hence the series converges a.e., so the terms converge to zero a.e. By the same argument

$$\frac{T^{n_k+\ell_k}g(x)}{\ell_k} \to 0 \quad \text{a.s.}$$

We now have convergence for f in a dense class of L^p. In [7] it is shown that for T induced by a measure preserving point transformation τ, T admits a dominated estimate on L^p along any sequence $\{n_k, \ell_k\}$ satisfying the growth condition given there. Consequently, Theorem 2.1 implies that T admits a dominated estimate along $\{n_k, \ell_k\}$ if T is a positive invertible isometry. By using the techniques which were used to prove Corollaries 2.2, 2.3, and 2.4 from Theorem 2.1, we see that we have a dominated estimate if T is power bounded Lamperti, or is dominated by a positive contraction. We now have convergence on a dense set, and a maximal inequality. Thus the result follows by a standard application of Banach's principle. ∎

Remark 1: An example of a sequence that satisfies the hypothesis of Theorem 3.1 is the sequence $n_k = 2^{2^k}$, and $\ell_k = \sqrt{n_k}$. This should be compared with the result in [2] where it is shown that

$$\frac{1}{\sqrt{n}}\sum_{i=0}^{[\sqrt{n}]} T^{n+i}(f)(x)$$

fails to converge even for $f \in L^\infty$, and $T(f)(x) = f(\tau x)$ for τ an ergodic measure preserving point transformation.

Remark 2: Dunford–Schwartz operators are dominated by positive contractions on L^p for each $p > 1$, consequently for Dunford–Schwartz operators we have convergence of averages of the type considered in Theorem 3.1

Let $B_k = \{n_k, n_{k+1}, \dots, n_{k+\ell_k}\}$ denote a block of ℓ_k consecutive integers starting at n_k. Let $B = \bigcup_{k=0}^{\infty} B_k$. Assume that the growth condition

$$\ell_k \geq c\, n_{k-1} \tag{3.1}$$

and the disjointness condition

$$\ell_k < n_{k+1} - n_k \tag{3.2}$$

of [8] are satisfied. Following Bellow and Losert we will call the sequence formed by the elements of B, in the natural order, a block sequence. In [8] it is shown that if T is induced by a measure preserving point transformation, then the averages along a block sequence,

$$\frac{1}{|B\cap[1,N]|}\sum_{j\in B\cap[1,N]} T^j f(x),$$

converge a.e. for all $f \in L^1$. We can now give a partial generalization for operators on L^p, $1 < p < \infty$.

3.2 THEOREM: *Assume $p > 1$. Let T be an operator on L^p which is dominated by a positive contraction, or a power bounded Lamperti operator. Then:*
(1) *the averages along a block sequence,*

$$\frac{1}{|B\cap[1,N]|}\sum_{j\in B\cap[1,N]} T^j f(x),$$

converge a.e. for all $f \in L^p$,

and

(2) *the maximal function*

$$Mf(x) = \sup_N \left| \frac{1}{|B \cap [1,N]|} \sum_{j \in B \cap [1,N]} T^j f(x) \right|$$

is a bounded operator from L^p to L^p.

Proof: In the special case, T induced by a measure preserving point transformation, it is shown in [20] that the operator $Mf(x)$ is a bounded operator from L^p to L^p. (This also follows from the result in [8]. Bellow and Losert show that when T is induced by a measure preserving point transformation, convergence holds for all $f \in L^1$. Consequently the maximal function satisfies a weak type (1,1) inequality. By interpolation, this gives the necessary strong type inequality for $p > 1$.) By Corollary 2.4, we now have the same dominated estimate for more general T. The convergence result will follow if we have convergence on dense class. For the class $\{f : f = g - Tg + h,\ g \in L^\infty,\ h \text{ invariant}\}$, convergence follows as in the proof of Theorem 3.1. ∎

Remark: This result and Theorem 3.1 do not extend to L^1 without additional restrictions on the operator. Chacon [13] has constructed an example of a positive contraction on L^1 for which even the usual Cesaro averages diverge a.e. for $f \in L^1$. Thus the fact that we do not get a theorem for L^1 is not just a consequence of our methods. There is a significant difference between L^1 and L^p, $p > 1$. There are special cases where a transference argument can be made for weak type (1,1) inequalities, but Chacon's example shows it cannot be done for all positive L^1 contractions.

In [6] it was shown that given $p_0 > 1$ a block sequence can be perturbed in such a way as to preserve a.e. convergence for $f \in L^p$, $p \geq p_0$, but such that convergence fails for some $f \in L^r$, for each $r < p_0$. This result can be generalized to the types of operators considered above.

3.3 COROLLARY: *Let B_k denote a block of consecutive integers satisfying conditions (3.1) and (3.2). For each k, let D_k denote a set of d_k integers between B_k and B_{k+1} and let $D = \bigcup_{k=1}^{\infty} D_k$. If*

$$\sum_{k=1}^{\infty} \left(\frac{d_1 + d_2 + \cdots + d_k}{\ell_1 + \ell_2 + \cdots \ell_k}\right)^{p_0} < \infty$$

then:

(1) *the maximal function associated with the set $D \cup B$ is bounded from L^p to L^p for each $p \geq p_0$,*

and

(2) *the associated limit exists a.e. for all $f \in L^p$, $p \geq p_0$.*

Proof: In [6] we are shown that for T induced by a measure preserving point transformation, convergence holds for $f \in L^p$, $p \geq p_0$. However this only gives a weak type inequality for L^{p_0} (hence a strong type inequality for $p > p_0$). To obtain a dominated theorem for L^{p_0} we proceed as follows. Let $S = D \cup B$. The maximal function

$$Mf(x) = \sup_N \left| \frac{1}{|S \cap [1,N]|} \sum_{j \in S \cap [1,N]} T^j f(x) \right|$$

can be dominated by the sum of two related maximal functions. Define

$$M_1 f(x) = \sup_N \left| \frac{1}{|S \cap [1,N]|} \sum_{j \in B \cap [1,N]} T^j f(x) \right|$$

and

$$M_2 f(x) = \sup_N \left| \frac{1}{|S \cap [1,N]|} \sum_{j \in D \cap [1,N]} T^j f(x) \right|.$$

Note that $\|Mf\|_{p_0} \leq \|M_1 f\|_{p_0} + \|M_2 f\|_{p_0}$, and that $M_1 f(x)$ is smaller than the maximal function for a block sequence. Therefore, as in Theorem 3.2 above, we have $\|M_1 f\|_p \leq C\|f\|_p$ for all $p > 1$, and in particular for p_0. To study $M_2 f(x)$ we use a different argument. Let $S_k = \bigcup_{j=1}^k D_j$, $\lambda_k = \sum_{j=1}^k \ell_j$, and $\delta_k = \sum_{j=1}^k d_j$. Define

$$A_k f(x) = \frac{1}{\lambda_k} \sum_{j \in S_k} T^j f(x).$$

Note that

$$\|A_k f\|_p \leq \frac{1}{\lambda_k} \sum_{j \in S_k} \|T^j f\|_p \leq \frac{1}{\lambda_k} \sum_{j \in S_k} C\|f\|_p \leq C\frac{\delta_k}{\lambda_k}\|f\|_p.$$

Also note that $M_2f(x) \le \sup_k |A_kf(x)|$. Consequently

$$\begin{aligned}
\|M_2f\|_p^p &\le \|\sup_k |A_kf|\|_p^p \\
&\le \|(\sum_{k=1}^{\infty} |A_kf|^p)^{1/p}\|_p^p \\
&\le \int \sum_{k=1}^{\infty} |A_kf(x)|^p dx \\
&\le \sum_{k=1}^{\infty} \int |A_kf(x)|^p dx \\
&\le \sum_{k=1}^{\infty} C^p(\frac{\delta_k}{\lambda_k})^p \|f\|_p^p \\
&\le C^p \|f\|_p^p \sum_{k=1}^{\infty} (\frac{\delta_k}{\lambda_k})^p.
\end{aligned}$$

By the hypothesis, if $p \ge p_0$ then the sum is finite, and we have the desired estimate for T induced by a measure preserving point transformation. Thus by Corollary 2.4 we have the maximal inequality for the more general operators. To obtain the necessary convergence on a dense subset, we consider as usual $\{f : f = g - Tg + h,\ g \in L^\infty,\ h \text{ invariant}\}$. We already know we have convergence on the block sequence portion of our sequence. The above argument and the Borel–Cantelli lemma show that for $p \ge p_0$, $A_kf(x) > \epsilon$ for only finitely many k. From this we see that convergence holds. ∎

4. Dunford–Schwartz Operators

Let $\{n_k\}$ be an increasing sequence of positive integers, and let T be an operator of L^p, $1 \le p \le \infty$. We will say that $\{n_k\}$ is a good sequence for T in L^p if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} T^{n_k}(f)(x) \text{ exists a.s.}$$

for all $f \in L^p$. (Recall that the sequence $\{n_k\}$ corresponds to the sequence $n(k,j) = n_j$ for all k in the notation of Section 2.)

In this section we will show that if $\{n_k\}$ is a good sequence for T in L^p, for all $p, 1 < p < \infty$, and all T induced by measure preserving point transformations, then each Dunford–Schwartz operator satisfies a dominated estimate in L^p, $1 <$

$p < \infty$, along $\{n_k\}$, and that for all Dunford–Schwartz operators, convergence holds for $f \in L^p$. This will show that the sequences considered by Bourgain and Wierdl are good in L^p, $1 < p < \infty$, for all Dunford–Schwartz operators. Bourgain, in [10], [11], and [12], and Wierdl [29] show that certain sequences are good in L^p, $1 < p \leq \infty$, for T induced by measure preserving point transformations, and that these operators admit a dominated estimate along the sequences considered. This implies by Corollary 2.4 that these sequences are good in L^p, $1 < p \leq \infty$ for Dunford–Schwartz operators.

4.1 THEOREM: *Let $\{n_k\}$ be a sequence that is good for all operators on L^p, $1 < p < \infty$, which are induced by one-to-one measure preserving point transformations, then $\{n_k\}$ is good in L^p for all Dunford–Schwartz operators.*

Proof: First note that since we have convergence for $f \in L^p, 1 < p < \infty$, for all T induced by one to one measure preserving point transformations, we also have for each fixed $p > 1$ and all such T a dominated estimate in L^p. Hence by Corollary 2.4, all Dunford–Schwartz operators admit a dominated estimate in L^p along $\{n_k\}$. Consequently it is enough to prove convergence on a dense subspace of L^p. To do this we adapt the proof of Theorem 2.19 of [4].

(i) The sequence $\{n_k\}$ is good for all operators induced in L^p spaces by arbitrary, but not necessarily invertible, measure preserving transformations. This follows in the usual manner by building two-sided shifts from one-sided shifts, and building one-sided shifts from measure preserving point transformations.

(ii) The sequence $\{n_k\}$ is good in L^p for all operators of the sort $T(f)(x) = v(x)f(\tau x)$, where $v \in L^\infty$, $0 \leq v \leq 1$, and τ is a measure preserving point transformation.

To see this, we will show that we have convergence along $\{n_k\}$ for f bounded. This will prove the theorem because we then have convergence on a dense subset of L^p, and we already have the associated dominated estimate for T along $\{n_k\}$. Thus the result will follow by a standard application of Banach's principle.

To obtain convergence on the dense set, let $I_n = [0,1]$, and let λ_n denote Lebesgue measure on I_n. Define $Y = \prod_{n=0}^{\infty} I_n$ and $\lambda = \prod_{n=0}^{\infty} \lambda_n$. Let ξ_n be the nth coordinate function on Y. Define $\varphi : X \times Y \to \{0, 1, 2, \ldots\}$ by

$$\varphi(x,y) = \min\{n : \xi_n(y) > v(\tau^n x)\}$$

with the convention that $\min\{\emptyset\} = +\infty$. Then

$$\lambda(\{y : \varphi(x,y) > n\}) = \prod_{k=0}^{n} v(\tau^k x),$$

so

$$T^n(f)(x) = \int \chi_{\{(x,y):\varphi(x,y)>n-1\}}(y) f(\tau^n x) d\lambda(y)$$

and

$$\frac{1}{N}\sum_{k=0}^{N-1} T^{n_k}(f)(x) = \int \frac{1}{N}\sum_{k=0}^{N-1} \chi_{\{(x,y):\varphi(x,y)>n_k-1\}}(y) f(\tau^{n_k} x) d\lambda(y).$$

Let E denote $\{(x,y) : \varphi(x,y) < \infty\}$. Then for each $(x,y) \in E$ we are averaging the terms of a finite sequence, or a bounded convergent sequence. For $(x,y) \notin E$, convergence follows by assumption since $\chi_{\{(x,y):\varphi(x,y)>n_k-1\}}(y) = 1$ for all k, and τ is a measure preserving point transformation. The result follows by an application of the dominated convergence theorem.

(iii) The sequence $\{n_k\}$ is good for all operators on L^p of the sort $T(f)(x) = v(x)f(\tau x)$, where v takes on complex values, and $|v| \le 1$.

This follows from the standard trick of Ryll–Nardzewski [28]. Let $(Y, \mathcal{S}, \lambda) = [0, 2\pi)$ with normalized Lebesgue measure. Let

$$B = X \times Y, \quad \mathcal{B} = \mathcal{F} \times \mathcal{S}, \quad \nu = \mu \times \lambda.$$

Write $v(x) = r(x)e^{i\varphi(x)}$ and define $\sigma : B \to B$ by $\sigma(x,\theta) = (\tau(x),\ \varphi(x) + \theta)$, where $\varphi(x) + \theta$ is computed mod 2π. Define V on $L^p(B, \mathcal{B}, \nu)$ by $V(g)(x,\theta) = r(x)g(\sigma(x,\theta))$. If $f \in L^p(X, \mathcal{F}, \mu)$ put $g(x,\theta) = f(x)e^{i\theta}$. Then by (ii)

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} V^{n_k} g(x,\theta) \text{ exists } \ \nu \text{ a.e. } (x,\theta).$$

Since $V^n g(x,\theta) = e^{i\theta}T^n(f)(x)$, the theorem follows in this case.

(iv) Now let T be a Dunford–Schwartz operator on L^p. Then we know there exists a positive Dunford–Schwartz operator S on $L^p(X)$ such that $|Tf(x)| \le S|f(x)|$. Let C and D be the conservative and dissipative parts of the Hopf decomposition of X, for S considered as an operator on $L^1(X)$. (See [17] or [22].) For all $f \ge 0$, $f \in L^1(X)$, $\sum_{k=0}^{\infty} S^k f(x) < \infty$ on D, so

$$\lim_{n\to\infty} \left| \frac{1}{n}\sum_{k=0}^{n-1} T^{n_k} f(x) \right| \le \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} S^{n_k}|f(x)| = 0$$

for all $f \in L^1(X)$, and in particular, all $f \in L^p(X)$, $p > 1$. If $f \in L^1(C)$, $Tf \in L^1(C)$ by Theorem 1.8 of [22, page 118]. Further, since S is an $L^\infty(X)$ contraction, $S(1) \leq 1$, and in particular $S(1_C) = 1_C$. Using the fact that X is a Lebesgue space, and $\mathcal{F}$ the σ-algebra of Borel sets, there exists a substochastic kernel $P(x, A)$, $x \in X$, $A \in \mathcal{F}$ (i.e. $P(x, \cdot)$ is a subprobability for each $x \in X$ and $P(\cdot, A)$ is an $\mathcal{F}$ measurable function for every $A \in \mathcal{F}$) such that for every $A \in \mathcal{F}$, $P(x, A)$ is a version of $S(1_A)(x)$. (See [25, page 192].)

A similar argument shows the existence of a complex valued kernel $\mu(x, A)$ which is a version of $T(1_A)(x)$ and such that $|\mu(x, A)| \leq P(x, A)$ for all $x \in X$ and $A \in \mathcal{F}$. Put

$$g(x, y) = \frac{d\mu(x, \cdot)}{dP(x, \cdot)}.$$

We need to show that we can find a version of $g(x, y)$, say $G(x, y)$, which is $\mathcal{F} \times \mathcal{F}$ measurable. To see this we argue as in [15, page 617]. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of open sets forming a base for the open sets in the topological space X. For each n let $\mathcal{F}_n$ denote the σ-field generated by the sets $(A_1, A_2, \ldots, A_n)$. Then $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and $\lim_{n\to\infty} \mathcal{F}_n = \mathcal{F}$. Define

$$g_n(x, y) = \frac{\mu_x(B)}{P(x, B)} \quad \text{for } y \in B \in \mathcal{F}_n, \quad B \text{ an atom}, \quad P(x, B) \neq 0,$$

and

$$g_n(x, y) = 0 \quad \text{for } y \in B \in \mathcal{F}_n, \quad B \text{ an atom}, \quad P(x, B) = 0.$$

With this definition, each g_n is $\mathcal{F} \times \mathcal{F}_n$ measurable. For each x, by the martingale convergence theorem, $\lim_{n\to\infty} g_n(x, y)$ exists a.e. $P(x, dy)$ and equals $g(x, y)$. Define

$$G(x, y) = \begin{cases} \lim_{n\to\infty} g_n(x, y) & \text{if the limit exists and is finite,} \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $G(x, y)$ is $\mathcal{F} \times \mathcal{F}$ measurable and for each $x \in X$, $G(x, y) = g(x, y)$ a.e. $P(x, dy)$. The complex valued function $G(x, y)$ satisfies

$$Tf(x) = \int G(x, y) P(x, dy) f(y)$$

for each $f \in L^1(X)$, and hence $f \in L^p(X)$ for each $p > 1$. We also have $|G(x, y)| \leq 1$. To see this let $A_x = \{y : G(x, y) > 1\}$. Then

$$\begin{aligned} T(1_{A_x})(x) &= \int G(x, y) P(x, dy) 1_{A_x}(y) \\ &> \int_{A_x} P(x, dy) = P(x, A_x) \\ &= S(1_{A_x})(x), \end{aligned}$$

a contradiction.

Now let $\Omega = \prod_{n=0}^{\infty} X_n$, where each $X_n = X$, $\mathcal{H} = \mathcal{F} \times \mathcal{F} \times \mathcal{F} \times \cdots$. Denote by P_x the measure induced by defining

$$\begin{aligned} &P_x(A_0 \times A_1 \times A_2 \times \cdots \times A_n \times X \cdots) \\ &\qquad = \chi_{A_0}(x) \int_{A_1} \int_{A_2} \cdots \int_{A_{n-1}} P(x_{n-1}, A_n) P(x_{n-2}, dx_{n-1}) \cdots P(x, dx_1) \end{aligned}$$

and by P_μ the measure induced by

$$\begin{aligned} &P_\mu(A_0 \times A_1 \times A_2 \times \cdots \times A_n \times X \cdots) \\ &= \int_{A_0} P_x(A_0 \times A_1 \times A_2 \times \cdots \times A_n \times X \cdots) d\mu(x) \\ &= \int_{A_0} \int_{A_1} \int_{A_2} \cdots \int_{A_{n-1}} P(x_{n-1}, A_n) P(x_{n-2}, dx_{n-1}) \cdots P(x, dx_1) d\mu(x). \end{aligned}$$

Define $\theta : \Omega \to \Omega$ by $\theta(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$. We now have a Markov process $(\Omega, \mathcal{H}, P_\mu, \theta)$. Assume that P_μ is invariant under θ. Let $V = G(\xi_0, \xi_1)$, where ξ_n is the nth coordinate function on Ω. Define W on $L^p(\Omega, \mathcal{H}, P_\mu)$ by $Wg(\omega) = V(\omega) g(\theta(\omega))$. To show that $\{n_k\}$ is good for T it is enough to show convergence for bounded functions, which are dense in all L^p, $1 \leq p \leq \infty$. For such an f, let $g = f \circ \xi_0$. By (iii)

$$\frac{1}{N} \sum_{k=0}^{N-1} W^{n_k} g \quad \text{converges} \quad P_\mu \text{ a.s.}$$

However,

$$\begin{aligned} Tf(x) &= \int_X G(x, x_1) P(x, dx_1) f(x_1) \\ &= \int_\Omega V(x, x_1, x_2, \ldots) g(x_1, x_2, \ldots) dP_x \end{aligned}$$

and, more generally, letting $\omega = (x, x_1, x_2, \ldots)$, we have

$$\begin{aligned} T^n f(x) &= \int_X \cdots \int_X G(x, x_1) \cdots G(x_{n-1}, x_n) P(x_{n-1}, dx_n) \cdots P(x, dx_1) f(x_n) \\ &= \int_\Omega V(\omega) V(\theta\omega) \cdots V(\theta^{n-1}\omega) g(\theta^n \omega) dP_x \\ &= E_x(W^n g(\cdot)). \end{aligned}$$

Thus $T^n f(x) = E_x(W^n g)$, where E_x is the conditional expectation operator with respect to the field $\mathcal{F} \times X \times X \cdots$. The theorem now follows from the bounded convergence theorem for conditional expectations.

It remains to show that P_μ is invariant under θ. This is known, see for example [1], but the proof is included here for completeness. It is enough to show

$$\begin{aligned} P_\mu(\theta^{-1}(A_1 \times A_2 \times \cdots \times A_n \times X \cdots) &= P_\mu(X \times A_1 \times A_2 \times \cdots \times A_n \times X \cdots) \\ &= P_\mu(A_1 \times A_2 \times \cdots \times A_n \times X \cdots) \end{aligned}$$

but

$$\begin{aligned} &P_\mu(X \times A_1 \times A_2 \times \cdots \times A_n \times X \cdots) \\ &= \int_X P_x(X \times A_1 \times A_2 \times A_3 \times \cdots \times A_n \times X \cdots) d\mu(x) \\ &= \int_X \int_{A_1} \int_{A_2} \cdots \int_{A_{n-1}} P(x_{n-1}, A_n) P(x_{n-2}, dx_{n-1}) \cdots P(x, dx_1) d\mu(x) \\ &= \int_X \int_X P_{x_1}(A_1 \times A_2 \times A_3 \times \cdots \times A_n \times X \cdots) P(x, dx_1) d\mu(x). \end{aligned}$$

To finish the argument we need the following lemma.

4.2 LEMMA: *For any bounded integrable function f, we have*

$$\int_X \int_X f(x_1) P(x, dx_1) d\mu(x) = \int_X f(x) d\mu(x).$$

Proof: First assume that $f(x) = \chi_E(x)$ for some measurable set E. Then we have

$$\int_X \int_X \chi_E(x_1) P(x, dx_1) d\mu(x) = \int_X P(x, E) d\mu(x) = \mu(B).$$

From this we see that if $f(x) = \sum_{i=1}^n a_i \chi_{E_i}(x)$ then

$$\begin{aligned} \int_X \int_X f(x_1) P(x, dx_1) d\mu(x) &= \sum_{i=1}^n a_i \int_X \int_X \chi_{E_i}(x_1) P(x, dx_1) d\mu(x) \\ &= \sum_{i=1}^n a_i \mu(E_i) \\ &= \int_X f(x) d\mu(x). \end{aligned}$$

Finally, let f_n be a sequence of simple functions increasing to the non-negative function f. Then apply the monotone convergence theorem to prove the lemma.

∎

Writing $f(x_1) = P_{x_1}(A_1 \times A_2 \times A_3 \times \cdots \times A_n \times X \cdots)$, and using the lemma, we now have

$$\begin{aligned}
&\int_X \int_X P_{x_1}(A_1 \times A_2 \times A_3 \times \cdots \times A_n \times X \cdots) P(x, dx_1) d\mu(x) \\
&= \int_X \int_X f(x_1) P(x, dx_1) d\mu(x) = \int_X f(x) d\mu(x) \\
&= \int_X P_x(A_1 \times A_2 \times A_3 \times \cdots \times A_n \times X \cdots) d\mu(x). \\
&= \int_{A_1} \int_{A_2} \cdots \int_{A_{n-1}} P(x_{n-1}, A_n) P(x_{n-2}, dx_{n-1}) \cdots P(x_1, dx_2) d\mu(x_1) \\
&= P_\mu(A_1 \times A_2 \times \cdots \times A_n \times X \times \cdots)
\end{aligned}$$

as required. ∎

4.3 COROLLARY: *Let $n_k = k^2$ (or more generally, k^t, t an integer), or let n_k denote the kth prime. Let $p > 1$, and assume T is a Dunford–Schwartz operator. Then $\{n_k\}$ is good for L^p and T.*

Proof: Bourgain ([10], [11], [12]) has shown that for T induced by a measure preserving point transformation, T admits a dominated estimate in L^p along $\{n_k\}$ with $n_k = k^t$, t an integer. Wierdl [29] has shown the same thing for n_k the kth prime. Theorem 4.1 now gives the result. ∎

5. Concluding Remarks

We have shown that if any aperiodic isometry T admits a dominated estimate in L^p, $1 < p < \infty$, along $\{n_k\}$, then all positive invertible isometries do as well, as do Dunford–Schwartz operators. We have thus shown that if Dunford–Schwartz operators admit a dominated estimate along $\{n_k\}$, then if $\{n_k\}$ is a good sequence in L^p for all operators on L^p induced by measure preserving point transformations, then it is good in L^p, $1 < p \leq \infty$, for all Dunford–Schwartz operators. This, in particular, implies that the sequences considered by Bourgain and Wierdl are good for Dunford–Schwartz operators in L^p.

This paper leaves open the question if these sequences are good in L^p for positive invertible isometries or indeed positive contractions of L^p, p fixed, $1 < p < \infty$. In the light of the results of Section 2, what is needed is a dense subset of L^p such that Cesaro averages of iterates of T taken along the subsequence $\{n_k\}_{k=1}^{\infty}$

converge for f in this class. This question will be answered in a subsequent paper by the use of variational inequalities.

To use the results of this paper for other sequences $\{n_k\}$ we would need to show that $\{n_k\}$ is good for all operators of L^p, $1 < p < \infty$, induced by measure preserving transformations, and that for at least one such T that is aperiodic, T admits a dominated estimate in L^p along $\{n_k\}$. These two results would then imply $\{n_k\}$ is good for $L^p, 1 < p \leq \infty$, and all Dunford–Schwartz operators.

ACKNOWLEDGEMENT: The authors wish to thank Professor Alexandra Bellow of Northwestern University and Professor Isak Kornfeld of North Dakota State University for their enthusiastic encouragement of this research, and for many helpful conversations on the subject of this paper. The second- named author also wishes to thank Northwestern University and the Office of the Dean, College of Science and Mathematics, North Dakota State University, for direct support of his efforts.

References

1. J. Aaronson, M. Lin and B. Weiss, *Mixing properties of Markov operators and ergodic transformations, and ergodicity of Cartesian products*, Isr. J. Math. **33** (1979), 198–224.
2. M. A. Akcoglu and A. del Junco, *Convergence of averages of point transformations*, Proc. Am. Math. Soc. **49** (1975), 265– 266.
3. M. A. Akcoglu and L. Sucheston, *Dilations of positive contractions in L^p spaces*, Can. Math. Bull. **20** (1977), 285–292.
4. J. R. Baxter and J. H. Olsen, *Weighted and subsequential ergodic theorems*, Can. J. Math. **35** (1983), 145–166.
5. A. Bellow, *Ergodic properties of isometries in L^p spaces*, $1 < p < \infty$, Bull. Am. Math. Soc. **70** (1964), 366–371.
6. A. Bellow, *Perturbation of a sequence*, Advances in Mathematics, to appear.
7. A. Bellow, R. L. Jones and J. Rosenblatt, *Convergence of moving averages*, Ergodic Theory and Dynamical Systems **10** (1990), 43–62.
8. A. Bellow and V. Losert, *On sequences of density zero in ergodic theory*, Contemporary Mathematics **26** (1984), 49–60.
9. A. Bellow and V. Losert, *The weighted pointwise ergodic theorem and the individual ergodic theorem along subsequences*, Trans. Am. Math. Soc. **288** (1985), 307–345.
10. J. Bourgain, *On the maximal ergodic theorem for certain subsets of the integers*, Isr. J. Math. **61** (1988), 39–72.

11. J. Bourgain, *On the pointwise ergodic theorem on L^p for arithmetic sets*, Isr. J. Math. **61** (1988), 73–84.
12. J. Bourgain, *Pointwise ergodic theorems for arithmetic sets*, IHES Publication, October, 1989.
13. R. V. Chacon, *A class of linear transformations*, Proc. Am. Math. Soc. **15** (1964), 560–564.
14. I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, *Ergodic Theory*, Springer-Verlag, New York, 1982.
15. J. Doob, *Stochastic Processes*, Wiley, New York, 1953.
16. N. Dunford and J. Schwartz, *Linear Operators, I*, Wiley, New York, 1958.
17. S. R. Foguel, *The Ergodic Theory of Markov Processes*, Van Nostrand, New York, 1969.
18. P. R. Halmos, *Lectures on Ergodic Theory*, Math. Soc. Japan, 1956.
19. R. L. Jones, *Inequalities for pairs of ergodic transformations*, Radovi Matematicki **4** (1988), 55–61.
20. R. L. Jones, *Necessary and sufficient conditions for a maximal ergodic theorem along subsequences*, Ergodic Theory and Dynamical Systems **7** (1987), 203–210.
21. C. Kan, *Ergodic properties of Lamperti operators*, Can. J. Math. **30** (1978), 1206–1214.
22. U. Krengel, *Ergodic Theorems*, De Gruyter Studies in Mathematics, 6, New York, 1985.
23. J. Lamperti, *On the isometries of certain function spaces*, Pacific J. Math. **8** (1958), 459–466.
24. E. Lorch, *Spectral Theory*, Oxford University Press, New York, 1962.
25. J. Neveu, *Mathematical Foundations of the Calculus of Probability*, Holden-Day, San Francisco, 1965.
26. J. H. Olsen, *Dominated estimates of convex combinations of commuting isometries*, Isr. J. Math. **11** (1972), 1–13.
27. V. A. Rohlin, *On the fundamental ideas of measure theory*, Mat. Sb. **25**, 107–150; Am. Math. Soc. Transl. **71** (1952).
28. C. Ryll-Nardzewski, *Topics in ergodic theory*, in *Proceedings of the Winter School in Probability, Karpacz, Poland*, Lecture Notes in Mathematics **472**, Springer-Verlag, Berlin, 1975, pp. 131–156.
29. M. Wierdl, *Pointwise ergodic theorem along the prime numbers*, Isr. J. Math. **64** (1988), 315–336.